Information Theoretic Cut-set Bounds on the Capacity of Poisson Wireless Networks

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***Abstract*—We present a stochastic geometry based model, in which we investigate the fundamental limitations of wireless networks. We derive cut-set bounds on the information theoretic capacity of networks of arbitrary Poisson node density, size, power, bandwidth, and fading characteristics. In other words, we upper-bound the optimal performance in terms of capacity, under any communication scheme, that can be achieved between a subset of network nodes (contained in the cut) with all the remaining nodes. Additionally, we identify four different operating regimes, thus confirming previously known scaling laws (*e.g.*, in bandwidth and/or power limited wireless networks), and extending them with specific bounds. Finally, we use our results to provide specific numerical examples.**

I. Introduction

The investigation of fundamental capacity limits of multi-node wireless networks is an open problem in information theory which consistently attracts the attention of researchers in recent years, as it is a difficult question with great potential practical interest. A way to approach the problem is to study the more restricted situation where wireless nodes are placed according to some spatial distribution (usually 2-dimensional), and to assume some specific spatial propagation model. Furthermore, we can assume that we are interested in the limit where the number of nodes tends to infinity and study the scaling behavior of the total network capacity.

In this context, initial investigations focused on scaling laws in restricted communication strategies, such as multi-hopping [2], providing important insights on the fundamental limits of wireless networks. Several works studied information-theoretic scaling laws, independent from the communication strategy. However, results usually provide only an asymptotic order for the network capacity (*e*.g., [7], [8]). Most importantly, these results provided insights on cooperation schemes with almost optimal scaling behavior. For instance, [8] shows that *dense* (*i.e.*, of fixed area and increasing density) and *extended* (*i.e.*, of fixed density and increasing area) networks exhibit qualitatively different scaling behaviors with regards to the total network capacity (linear and sub-linear or square root increase with respect to the number of nodes, respectively). In contrast, real networks have a fixed area and density and, in this sense, such scaling laws are a paradox for practical purposes. This limitation has been partially addressed with an insightful extension in [7], where it is shown that important parameters defining the asymptotically optimal operating regime of a wireless network are the short range and long range signal to noise ratios.

In this paper, we focus on the derivation of fundamental upper bounds on the capacity of such wireless networks. In contrast to previous approaches, we rely on an infinite Poisson network model, which we analyze using a stochastic geometry methodology. Lately, this methodology has been used extensively to analyze the performance of wireless networks (see [3] for a recent survey). However, the full potential of the Poisson point process modeling has not been exploited as regards the derivation of information theoretic results. Therefore, this paper evaluates the cut-set capacity of cuts of arbitrary size, for arbitrary values of all the other network parameters, such as the node density, transmit power, channel bandwidth, noise spectral density. In other words, we bound the capacity, under any communication strategy, that is achievable between a subset of network nodes (contained in the cut) with all the remaining nodes. This is an interesting problem in its own right that has not been addressed in the literature. Moreover, cut-set bounds essentially constitute the only powerful tool that is currently available to bound wireless network capacities.

Additionally, instead of working with scaling laws, we calculate bounds on the expected cut-set capacities, averaged over all possible Poisson node configurations. Thus, we analyze a meaningful performance measure, such as the expected network capacity in the case where the nodes are distributed randomly or move slowly (but they remain Poisson distributed). This formulation allows us to derive specific numeric bounds, and to capture the continuous transitions between different operating regimes, complementing previous related work. But, we also analyze the asymptotic behavior of our bounds (depending on all the parameters), and we confirm that our results are in agreement with the previously known scaling laws, which can be mapped to different operating regimes (as identified in [7], [8]).

In Section II, we introduce our network model and we discuss our main results, corresponding to the following main contributions: (1) we use a stochastic geometry-based methodology in order to calculate a bound on MISO (Multiple Input Single Output) capacities in Section III; (2) we prove our general cut-set capacity bound (Theorem 1), in Section IV, we analyze the asymptotic behavior, and we identify different network operating regimes (Corollary 1); (3) we provide specific numerical examples in Section V.

II. Model and Main Results

Nodes are placed on an infinite plane according to a Poisson point process of intensity ν, and can transmit at maximum

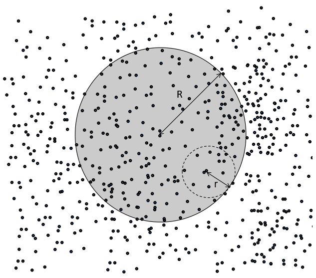

Fig. 1. Circular cut partitioning the infinite plane into a disk of radius R and the remaining region, and a node at distance r from the cut.

power P watts. We assume that node positions remain fixed during a channel use. The received power decays with the distance in power law, with path loss exponent $\alpha > 2$, and is subject to space-time *i.i.d.* random fading and shadowing, denoted with random variable g (where, *w*.l.o.g., $\mathbf{E}(g) = 1$); *i.e.*, for maximum transmit power, the received power at distance r from the transmitter equals $Pg \cdot r^{-\alpha}$. Therefore, our model cannot capture non-stationary fading variations with time, or the variation of the signal power over short distances of the order of the wavelength. The communication channel has bandwidth W hertz (the carrier frequency is supposed to be much larger than W). The background noise is additive white Gaussian (AWGN), with power spectral density $\frac{N}{2}$ watts per hertz, *i.e.*, the noise is *i.i.d*. Gaussian in space and time.

It is important to note that the *i.i.d.* fading assumption concerns only the factor g by which the power is multiplied; our model makes no assumption regarding the signal phases. This comes at the cost of less detailed modeling. However, in our analysis, this model generality makes our results stronger, since we are interested in the computation of upper bounds that hold even in the best-case scenarios.

A. Main Results

To motivate our approach, let us first consider a point-to-point AWGN channel. The channel capacity is given by the simple formula: $C = W \cdot \log\left(1 + \frac{P}{NW}\right)$, with P in this case being the received power. Moreover, we identify two operating regimes: for low SNR, we have $C \sim \frac{P}{N}$, and the capacity is power-limited; for high SNR, we have $C \sim W \log \frac{P}{NW}$, and the capacity is essentially bandwidth-limited.

In this paper, we provide such a unified formula, as an upper bound on cut-set capacities in Poisson wireless networks. We then show that its asymptotic behavior is richer, but not too complicated (providing four different asymptotic regimes).

We consider an approximately circular cut $\mathcal{D}_R$ of radius asymptotically equal to R, partitioning the plane into two regions $\mathcal{D}_R$ and $\mathcal{D}_R^C$, as depicted in Figure 1.

We compute an upper bound on the communication rate achievable between all nodes in $\mathcal{D}_R$ and all nodes in $\mathcal{D}_R^C$. So, we define the cut-set capacity C_R, which corresponds to the total capacity between all nodes in $\mathcal{D}_R$ and all nodes in $\mathcal{D}_R^C$, when all nodes at each side of the cut can cooperate fully. We evaluate the cut-set capacity $\bar{C}_R = \mathbf{E}(C_R)$, averaged over all possible node position configurations.

Theorem 1. *When $\nu R^2 \to \infty$, the expected cut-set capacity is bounded by $\bar{C}_R^* \leq \mathcal{I}^*$, and:*

$$\mathcal{I}^* \sim 2\pi\nu W \int_d^R \log\left(1 + s_r\right)(R - r)dr,$$

with $s_r = \frac{2\pi\nu r^{2-\alpha}}{\alpha - 2}\frac{P}{NW}$, and d the critical percolation radius.

Therefore, Theorem 1 provides a simple, unified and multi-parameter bound on the cut-set capacity. This cut-set bound corresponds to an ideal scenario, and always constitutes an upper bound, irrespective of the channel characteristics and available information; hence, it is not achievable in general.

The parameter $s_r = \frac{2\pi\nu}{\alpha-2} r^{2-\alpha} \frac{P}{NW}$, corresponds to an upper bound on the expectation of the total SNR received at a given location from all nodes at range at least r.

Hence, we can identify different asymptotic scaling laws, corresponding to the behavior of s_r at the upper ($r = R$) and lower ($r = d$) limits of the integral (tending to 0 or ∞). The asymptotic derivations are detailed in Corollary 1, in Section IV. Setting $n = \pi\nu R^2$ (the expected number of nodes in the cut) in the latter, we obtain the following asymptotic bounds (with a slight abuse of the notation $\sim$, we omit all the constants for simplicity):

$$\mathcal{I}^* \sim \begin{cases} Wn\log(\frac{P}{NW}), & s_R = \omega(1) \quad (I) \\ n^{2-\frac{\alpha}{2}}\frac{P}{N}, & s_R = o(1),\ \alpha < 3 \quad (II) \\ \sqrt{n}\left(\frac{P}{N}\right)^{\frac{1}{\alpha-2}} W^{\frac{\alpha-3}{\alpha-2}}, & —,\ \alpha > 3,\ s_d = \omega(1) \quad (III) \\ \sqrt{n}\frac{P}{N}, & —,\ —,\ s_d = o(1). \quad (IV) \end{cases}$$

In words, we identify four different asymptotic regimes. When $s_R = \omega(1)$, the upper bound indicates that the cut-set capacity is linear in n and bandwidth-limited (I). When $s_R = o(1)$, the capacity is power-limited and sub-linear in n when $\alpha \leq 3$ (II), and both power (long-range) and bandwidth (short-range) limited when $\alpha > 3$ and $s_d = \omega(1)$ (III). When $s_d = o(1)$, the power limitation dominates at all ranges (IV). In the two latter cases, the capacity bound is $\Theta(\sqrt{n})$.

Therefore, with s_R corresponding to the long-range SNR, and s_d to the short-range SNR, the different cases described above essentially map to the operating regimes identified in [7], derived under a different perspective and methodology (here, concerning expected cut-set capacities, with no network scaling arguments involved). Accordingly, even though we do not compute lower bounds, the relative asymptotic tightness of our bounds is established by comparing with these related results; the four optimal communication schemes discussed in [7] would achieve almost order-optimal scaling performance if analyzed in our framework (if the transmission medium is sufficiently rich, *e.g.*, in a random phase fading model).

III. MISO Bound

Consider a disk of radius $r > 0$, centered at the location of a node A. We denote Q_r the total received SNR by node A from all nodes outside the disk, *i.e.*, the total received power, normalized by the background noise NW. We compute an upper bound on the expectation $\mathbf{E}(Q_r)$ over all possible node configurations.

Lemma 1. *The expectation $\mathbf{E}(Q_r)$ of the total received SNR from all nodes outside a disk of radius $r > 0$ is at most:*

$$s_r = \frac{2\pi\nu r^{2-\alpha}}{\alpha - 2}\frac{P}{NW}. \tag{1}$$

Proof: Assuming that all nodes transmit at maximum power P, the expectation can be directly computed by using Campbell's theorem [4, p. 28]:

$$\mathbf{E}(Q_r) \leq \mathbf{E}_g \int_0^{2\pi} d\phi \int_r^{\infty} \rho d\rho \cdot \nu \rho^{-\alpha} \frac{Pg}{NW}, \tag{2}$$

applied to the marked Poisson process, generated by the node positions and the *i.i.d.* power fading g. ■

The latter upper bound is valid independently of the communication strategy that will be eventually employed and of the actual fading, as it is based on the fundamental power conservation limitation. Therefore, we can similarly formulate a fundamental upper-bound on the expected Multiple Input Single Output (MISO) capacity, when all nodes outside the disk of radius r transmit to node A at its center.

Lemma 2. *The expectation over all Poisson node configurations of the MISO capacity is:*

$$\mathbf{E}(C_r^{MISO}) \leq W \log\left(1 + s_r\right). \tag{3}$$

Proof: From the formula for the AWGN MISO capacity, with bandwidth W and total received SNR Q_r [1, Ch. 15.3]:

$$C_r^{\text{MISO}} = W \log(1 + Q_r). \tag{4}$$

As $\log(1+x)$ is concave, we conclude using Jensen's inequality, *i.e.*, $\mathbf{E}(\log(1 + Q_r)) \leq \log(1 + \mathbf{E}(Q_r)) \leq \log(1 + s_r)$. ■

IV. Cut-set Capacity: Proof of Theorem 1

We consider a cut partitioning the plane into two regions $\mathcal{D}$ and $\mathcal{D}^C$. We compute an upper bound on the communication rate achievable between all nodes in $\mathcal{D}$ and all nodes in $\mathcal{D}^C$, by defining the cut-set capacity $C^{\mathcal{D}}$. Let $Y^{\mathcal{D}}$ the vector of signals received from all nodes at one side of the cut (after attenuation, fading and Gaussian noise addition), and $X^{\mathcal{D}^C}$ be the vector of signals sent from all nodes at the opposite side. The cut-set capacity equals the maximum of the mutual information [1, Ch. 15.10], over all possible distributions of $X^{\mathcal{D}^C}$ satisfying the maximum power constraint, *i.e.*,

$$C^{\mathcal{D}} = \max(\mathcal{I}(X^{\mathcal{D}^C}; Y^{\mathcal{D}})). \tag{5}$$

Lemma 3. *The cut-set capacity is upper bounded by:*

$$C^{\mathcal{D}} \leq \int_{\mathcal{D}} \nu dS \cdot C_{MISO}(z), \tag{6}$$

where $C_{MISO}(z)$ is the MISO capacity from all nodes at the opposite side of $\mathcal{D}$ to a node at the location z of dS.

Proof: If we divide $\mathcal{D}$ into two regions $\mathcal{C}$ and $\mathcal{C}^C$:

$$\begin{aligned}\mathcal{I}(X^{\mathcal{D}^C}; Y^{\mathcal{D}}) &= \mathcal{I}(X^{\mathcal{D}^C}; Y^{\mathcal{C}}, Y^{\mathcal{C}^C}) \\ &\leq \mathcal{I}(X^{\mathcal{D}^C}; Y^{\mathcal{C}}) + \mathcal{I}(X^{\mathcal{D}^C}; Y^{\mathcal{C}^C}),\end{aligned}$$

as the space components of the AWGN channel noise received at $\mathcal{C}$ and $\mathcal{C}^C$ are independent. Generalizing from two to several partitions, and taking the limit to partitions dS of $\mathcal{D}$ (which, as we show below, eventually include at most one node each), we combine with (5) to obtain the cut-set capacity bound:

$$C^{\mathcal{D}} \leq \int_{\mathcal{D}} C_{dS}(z), \tag{7}$$

where $C_{dS}(z)$ is the capacity between all nodes at the opposite side of the cut $\mathcal{D}$ and nodes in the area dS, located at z.

Since the distribution of the nodes is Poisson, the probability $P(k)$ to have exactly k nodes in the area dS is location-invariant, and equals $P(k) = (\nu dS)^k \frac{e^{-\nu dS}}{k!}$. Therefore,

$$P(k) = \begin{cases} 1 - \nu dS + O(dS^2), & k = 0 \\ \nu dS + O(dS^2), & k = 1 \\ O(dS^2), & \text{otherwise.} \end{cases}$$

As a result, at the limit, the capacities $C_{dS}(z)$ become equal to the MISO capacities $C_{\text{MISO}}(z)$ between a node at the location z of dS and the nodes at the other side of the cut, multiplied by the probability of presence (νdS) of a node in dS, *i.e.*,

$$\int_{\mathcal{D}} C_{dS}(z) = \int_{\mathcal{D}} \nu dS \cdot C_{\text{MISO}}(z). \tag{8}$$

This analysis holds even if the capacities $C_{\text{MISO}}(z)$ are dependent random variables for different values of z (but, the MISO capacities are not affected by the presence of nodes in the region $\mathcal{D}$, due to the node distribution being Poisson). ■

We now consider a circular cut $\mathcal{D}_R$ of radius R, as depicted in Figure 1, and we assume the existence of an empty outer strip of width d. We evaluate the cut-set capacity $\bar{C}_R = \mathbf{E}(C_R)$, averaged over all node position configurations.

Lemma 4. *The cut-set capacity is $\bar{C}_R \leq \mathcal{I}$, with:*

$$\mathcal{I} = 2\pi\nu W \int_d^R \log\left(1 + s_r\right)(R - r) dr. \tag{9}$$

Proof: We define $c(r)$ as the MISO capacity of a node located in the disk $\mathcal{D}_R$, at distance r from the border. From Lemma 3, and the linearity of expectations (*i.e.*, $E(X+Y) = E(X) + E(Y)$, even if X and Y are dependent),

$$\begin{aligned}\bar{C}_R &\leq \mathbf{E} \int_d^{2\pi} d\phi \int_0^R (R - r) dr \cdot \nu c(r) \\ &= 2\pi\nu \int_d^R (R - r) dr \cdot \mathbf{E}(c(r)).\end{aligned} \tag{10}$$

We have: $\mathbf{E}(c(r)) \leq \mathbf{E}(C_r^{\text{MISO}})$, as the capacity C_r^{MISO} corresponds to a smaller cut of radius r, contained inside $\mathcal{D}_R$ (see Figure 1). Finally, we substitute Lemma 2 into (10). ■

Lemmas 3 and 4 allow us to use the MISO bounds in order to obtain upper bounds on cut-set capacities. Our bounding approach is unhindered by the dependence between received SNRs at different nodes, as it is based on fundamental power limitations, but there are no additional assumptions regarding signal fading. In this sense, it captures all the diversity of the wireless medium (implicitly assuming that there are as

many degrees of freedom as can be used). Thus, it is valid for any communication strategy and any actual fading. In contrast, in [8], the MIMO-based upper bound required a complicated eigenvalue analysis to show that sending *i.i.d.* signals is scaling optimal (in the random phase model).

A. Approximately Circular Cut

We let the cut become large, such that the expected number of nodes in the disk tends to infinity, *i.e.*, $\nu R^2 \to \infty$. In the following lemma, using percolation theory, we show that there exists indeed a cut of radius approximately R, with an empty outer strip of width $d = \frac{r_c}{\sqrt{\nu}}$, where r_c is the critical percolation radius for unit node density.

Lemma 5. *For some constant $\rho > 0$, the annulus $\mathcal{A}(R, R + \rho \log(\sqrt{\nu}R))$ contains almost surely (when $\nu R^2 \to \infty$) a vacant loop of width $\frac{k}{\sqrt{\nu}}$, for any constant $k < r_c$.*

Proof: See appendix. ■

To complete the proof of Theorem 1, we bound the expected cut-set capacity of the approximately circular cut $\mathcal{D}_R^*$.

Proof: Since Lemma 5 holds for any $k < r_c$, we can assume that the smaller distance between two nodes at opposite sides of the cut is exactly $d = \frac{r_c}{\sqrt{\nu}}$. The cut-set capacity upper bound can be computed from Lemma 4. The larger distance between opposite side nodes is $R + \rho \log(\sqrt{\nu}R) \sim R$. Therefore, it can be verified that the integral remains asymptotically equivalent if we take R as the upper limit in Lemma 4. ■

Corollary 1. *When $\nu R^2 \to \infty$, there is an approximately circular cut, of radius $R^* \sim R$, such that the cut-set capacity is upper bounded by $C_R^* \leq \mathcal{I}^*$, with:*

$$\mathcal{I}^* \sim \begin{cases} \pi \nu R^2 W \log(\frac{\nu R^{2-\alpha} S}{W}), & s_R = \omega(1) \\ K_1 \cdot \nu^2 R^{4-\alpha} S, & s_R = o(1),\ \alpha < 3 \\ 4\pi^2 \nu^2 R \log(R) S, & —,\ \alpha = 3 \\ K_2 \cdot \nu^{\frac{\alpha-1}{\alpha-2}} R S^{\frac{1}{\alpha-2}} W^{\frac{\alpha-3}{\alpha-2}}, & —,\ \alpha > 3,\ s_d = \omega(1) \\ K_3 \cdot \nu^{\frac{1+\alpha}{2}} R S, & —,\ —,\ s_d = o(1), \end{cases}$$

with $K_1 = \frac{4\pi^2}{(\alpha-2)(3-\alpha)(4-\alpha)}$, $K_2 = \frac{(2\pi)^{\frac{\alpha-1}{\alpha-2}}}{(\alpha-2)^{\frac{1}{\alpha-2}}} \frac{\pi}{\sin(\frac{\pi}{\alpha-2})}$, $K_3 = \frac{4\pi^2}{(\alpha-2)(\alpha-3)} r_c^{3-\alpha}$.

Proof: See appendix. ■

V. Numerical Results

We provide numerical examples, illustrating the derived cut-set bounds. Figure 2 depicts log-log plots of the bounds on the expected cut-set capacity $\bar{C}_R^*$ in bits per second, by varying the signal to noise parameter $\frac{P}{NW}$, for $\alpha = 2.5$ (top) and $\alpha = 4$ (bottom). The remaining parameters are fixed: density $\nu = 1$, radius $R = 100$, bandwidth $W = 10^3$. We use the numeric estimate $d = 1.198$ for the percolation radius. The solid lines plot the upper bound from Theorem 1. The dashed lines are the asymptotic bounds in Corollary 1 (for case (I), we add the second-order constant factor from (11) in the proof). Insets plot the long-range (s_R) and short-range (s_d) SNRs.

The figures illustrate the continuous transitions between the four different operating regimes we described in Section II-A.

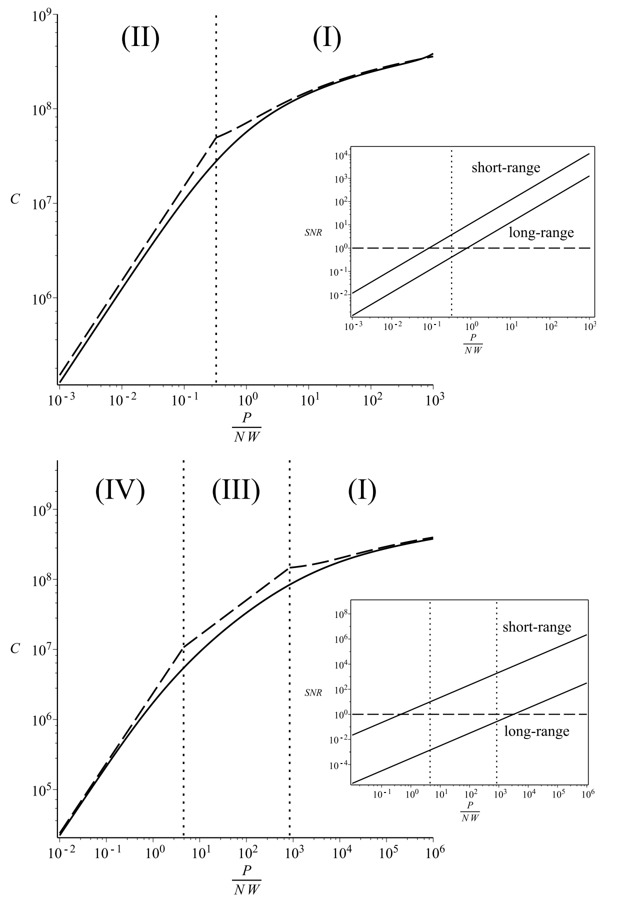

Fig. 2. Cut-set capacity bound C, versus the SNR parameter $\frac{P}{NW}$, for $\alpha = 2.5$ (top) and $\alpha = 4$ (bottom), and corresponding asymptotic regimes. Small graphs plot the long-range and short-range SNRs.

When $\alpha < 3$, we have two different operating regimes (Figure 2, top). When the long-range SNR is small, the cut-set capacity increases linearly, corresponding to the power-limited regime (II). When the long-range SNR becomes larger, we observe a slower logarithmic increase, and we identify the bandwidth-limited regime (I). In contrast, when $\alpha > 3$, the evolution of the SNR reveals three operating regimes (Figure 2, bottom). When both the short-range and long-range SNRs are small, the linear capacity growth indicates the power-limited regime (IV); then, when the short-range SNR becomes large but the long-range SNR is still small, the slope changes and the regime is both power and bandwidth-limited (III); finally, when the long-range SNR becomes large too, the capacity growth slows down considerably, as we transition to the bandwidth limited regime (I).

VI. Conclusion

We derived a new unified multi-parameter upper bound on the cut-set capacity of wireless networks. The asymptotic analysis reveals four operating regimes, which can be mapped to previously known scaling laws [7], extending them with specific bounds. Our methodology provides an analytically tractable framework; therefore, possible future extensions in-

clude tightening our analysis, and providing lower bounds in specific channel models.

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VII. APPENDIX

A. Proof of Corollary 1

The integral in Theorem 1 equals $\mathcal{I} = 2\pi\nu W \cdot I_r|_d^R$, where $I_r = \int \log(1+s_r)(R-r)dr$ can be evaluated in closed form (as can be easily checked using Maple or Mathematica). We initially assume that $\{\frac{1}{\alpha-2}, \frac{2}{\alpha-2}\} \neq 1, 2, \ldots$, to obtain a general formula. This excludes $\alpha = 3, 4$, while all other excluded values are smaller than 3.

We recall the definition of ordinary hypergeometric functions [6, Ch. 15]: ${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$, $c \neq 0, -1, -2, \ldots$, and $(x)_n = x(x+1)\ldots(x+n-1)$ is the rising factorial (with $(x)_0 = 1$). We have:

$$\begin{aligned} I_r = {} & \log(1+s_r)\left(R - \frac{r}{2}\right) r + (\alpha-2)\left(R - \frac{r}{4}\right) r \\ & - (\alpha-2)\left(Rg_1(s_r) - \frac{r}{4}g_2(s_r)\right) r, \end{aligned} \tag{11}$$

with $g_k(x) = {}_2F_1\left(1, -\frac{k}{\alpha-2}; 1 - \frac{k}{\alpha-2}; -x\right), k = \{1, 2\}$.

The definitions of $g_k(s_r)$ as hypergeometric functions yield full asymptotic expansions for $s_r = o(1)$.

Using a linear transformation [6, eq. 15.8.2], we obtain full asymptotic expansions for $s_r = \omega(1)$:

$$\begin{aligned} g_k(s_r) = {} & \frac{1}{s_r}\frac{k \cdot {}_2F_1\left(1, 1 + \frac{k}{\alpha-2}; 2 + \frac{k}{\alpha-2}; -\frac{1}{s_r}\right)}{k + \alpha - 2} \\ & + s_r^{\frac{k}{\alpha-2}} \frac{k\pi}{(\alpha-2)\sin(\frac{k\pi}{\alpha-2})}, \quad k = \{1, 2\}). \end{aligned} \tag{12}$$

Combining all these expansions, the asymptotic analysis of the integral I_r is straightforward. The main asymptotic terms for both integration limits are:

$$I_R \sim \begin{cases} \frac{R^2}{2}\log(s_R), & s_R = \omega(1) \\ \frac{2\pi\nu R^{4-\alpha}}{(\alpha-2)(3-\alpha)(4-\alpha)}\frac{P}{NW}, & s_R = o(1) \end{cases} \tag{13}$$

$$I_d \sim \begin{cases} Rs_1^{\frac{1}{\alpha-2}}\frac{\pi}{\sin(\frac{\pi}{\alpha-2})}, & s_d = \omega(1) \\ \frac{2\pi\nu R d^{3-\alpha}}{(\alpha-2)(3-\alpha)}\frac{P}{NW}, & s_d = o(1). \end{cases} \tag{14}$$

When $s_R = \omega(1)$, the main asymptotic term is always I_R, *i.e.*, the first case of (13).

When $s_R = o(1)$ and $\alpha < 3$, the main asymptotic term is again I_R, now equal to the second case in (13).

When $s_R = o(1)$ and $\alpha > 3$, the main asymptotic term is I_d. So, we use the two cases of (14), when $s_d = \omega(1)$ and $s_d = o(1)$, respectively.

For completeness, we consider the excluded values of α. For $\alpha = 3$, $\alpha = 4$, a simple integration confirms Corollary 1. For the remaining cases, we have $\alpha < 3$, and the main asymptotic terms are the same as in the general case. It suffices to note that, when $s_R = \omega(1)$, we can use $\log(1+s_r) = \log(s_r) + o(1)$ to recover the main asymptotic term: $\pi\nu R^2\log(s_R) + O(\nu R^2)$. When $s(R) = o(1)$, we use the fact that $\log(1+s_r) \leq s_r$ to perform the integration on s_r. The result is asymptotically tight; for $\alpha < 3$, the upper limit I_R is always dominant, and indeed $\log(1+s_r) = s_r + o(s_r)$ when $r = \Theta(R)$.

B. Proof of Lemma 5

We consider the Boolean continuum percolation model [5] where nodes are placed with Poisson intensity ν, and they are connected within distance x. The critical percolation radius with node density ν is $\frac{r_c}{\sqrt{\nu}}$, where r_c is the critical percolation radius with unit node density.

We follow the definition of vacant and occupied regions from [5, p. 15]. We consider an annulus of inner perimeter ℓ and width m. Let $P(\text{vacant-loop})$ be the probability that the annulus contains a vacant loop of width x. Let $P(\text{TB-occupied})$ be the probability that the annulus contains an occupied top-bottom crossing connecting the two circular sides. Clearly,

$$P(\text{vacant-loop}) = 1 - P(\text{TB-occupied}). \tag{15}$$

Let P_i, with $i \in \mathbb{Z}$, $0 \leq i < \frac{\ell}{x}$ be a sequence of points on the inner annulus boundary, at equal distance x (except possibly the two points closing the circle, which are at distance at most x). For the existence of an occupied component in the direction of width m, there must be at least one occupied component of diameter at least m, from some P_i. The probability that a connected component of diameter at least m exists, is bounded by the location invariant probability that there is a connected path from the origin to the boundary of a square box $[-m, m] \times [-m, m]$, centered at the origin, which we denote: $P(0 \overset{o}{\rightsquigarrow} \partial B_m)$. Taking a union bound:

$$P(\text{TB-occupied}) \leq \frac{\ell}{x} \cdot P(0 \overset{o}{\rightsquigarrow} \partial B_m). \tag{16}$$

For any $x = \frac{k}{\sqrt{\nu}}$, where k is a constant independent of ν such that $k < r_c$, Theorem 2.4 in [5] implies that:

$$P(0 \overset{o}{\rightsquigarrow} \partial B_m) \leq K_1 e^{-K_2 m}, \tag{17}$$

for some constants $K_1, K_2 > 0$ depending on k.

Therefore, from (15) and (16), we obtain the bound:

$$P(\text{vacant-loop}) \geq 1 - K_1\frac{\ell\sqrt{\nu}}{k}e^{-K_2 m}. \tag{18}$$

Taking $\ell = 2\pi R$, and $m = \rho\log(R\sqrt{\nu})$, with $\rho > \frac{1}{K_2}$,

$$P(\text{vacant-loop}) \geq 1 - \frac{K_1}{k}\left(R\sqrt{\nu}\right)^{1-\rho K_2} \underset{\nu R^2 \to \infty}{\to} 1. \tag{19}$$